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Siegel disc singularity spectra

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Abstract. We numerically determine the generalized dimensions and spectrum of singularities of a critical point orbit on the boundary of a Siegel disc. The local scaling exponent, α_{MN}, and spectrum dependence on the degree, d, of the critical point are also considered. We find that $\lim_{d\to\infty}|\alpha_{\mathrm{MN}}|^d \approx 0.523\,53$ and hence, asymptotically, the lower limit of the spectrum is constant whilst the upper limit is proportional to d.

1. Introduction

Recently, Halsey *et al* [1] have introduced a global description of certain strange sets arising in dynamical systems by considering the singularities of measures associated with the system (see also [2]). They define a 'spectrum of singularities', giving a set of possible singularity strengths, α, and corresponding Hausdorff dimensions $f(\alpha)$. The indices α and the function $f(\alpha)$ generalize previously defined characteristics such as local scaling indices and Hausdorff dimension. One of the examples Halsey *et al* study is the critical orbit at the onset of chaos in circle maps with golden mean rotation number. They produce a smooth function $f(\alpha)$, defined on an interval $[\alpha_{\min}, \alpha_{\max}]$. The maximum of this function is the Hausdorff dimension of the support of the associated measure (1 in this case) and $\alpha_{\min}$ and $\alpha_{\max}$ are simply related to the universal local scaling behaviour in the vicinity of the point of inflexion of the circle map as found by Shenker [3]. The universality of this formalism has been supported by experimental results on fluid convection and electronic transport phenomena [4].

The $f(\alpha)$ spectrum has also been studied in other critical quasiperiodic systems. Tang and Kohmoto [5] study the spectrum of a quasiperiodic Schrödinger operator and Osbaldestin and Sarkis [6] study KAM tori in area-preserving twist maps.

In this paper, we consider another quasiperiodic orbit, namely the critical point orbit on the boundary of a Siegel disc. The complex scaling exponent $\alpha_{\mathrm{MN}} \approx -0.22027 + 0.70848\mathrm{i}$ ($|\alpha_{\mathrm{MN}}| \approx 0.741\,932$) in the vicinity of the critical point has been identified by Manton and Nauenberg [7] for maps of the form $f(z) = \lambda z + \mathrm{O}(z^2)$, with $\lambda = \exp(2\pi\mathrm{i}\gamma)$, $\gamma = \frac{1}{2}(\sqrt{5}-1)$, the golden mean. Widom [8] has also given a renormalization group analysis of this scaling behaviour. As in the circle map case of Halsey *et al*, we find a smooth function $f(\alpha)$ defined on a theoretically predicted range $[\alpha_{\min}, \alpha_{\max}]$.

We also investigate the dependence of the singularity spectrum on the degree, d, of the critical point, having first presented the degree dependence of the local scaling exponent α_{MN}. Several groups ([9–11] amongst others) have considered the degree dependence of the analogous exponent for period doubling cascades. The corresponding circle map problem has also been widely studied [12]. Recently Hu *et al* [13] have investigated the degree dependence of the exponents of KAM tori in area-preserving twist maps. The degree dependence of the dimensions in the period doubling scenario have been considered by Hu and Mao and by Ambika and Joseph [14] (see also [10]). The degree dependence of the dimension associated with the mode-locking intervals in circle maps has been explored by Alstrøm and by Delbourgo and Kelly [15].

Here we find that $\alpha_{\rm MN} \to 1$ as $d \to \infty$. Investigation of the form of the approach reveals that $\lim_{d\to\infty} |\alpha_{\rm MN}|^d \approx 0.523\,53$. An identical asymptotic form for the period doubling scenario has been proved by Eckmann and Wittwer [11] (see also van der Weele *et al* [10]). For the singularity spectrum this result implies that, asymptotically, $\alpha_{\rm min}$ is independent of degree, whilst $\alpha_{\rm max}$ is proportional to d.

2. Siegel disc singularity spectrum

The quadratic complex map

$$f(z) = \frac{\lambda}{2} z^2 + 1 - \frac{\lambda}{2} \tag{2.1}$$

with $\lambda = \exp(2\pi i\gamma)$, $\gamma = \frac{1}{2}(\sqrt{5} - 1)$, has a fixed point at $z = 1$, with linearization pure rotation by $2\pi\gamma$, and a critical point, z_0, at $z = 0$. The iterates of the critical point, $\{z_0, z_1 = f(z_0), z_2 = f(f(z_0)), \ldots\}$, lie on a fractal curve (the boundary of the Siegel disc) and it is their distribution that the singularity spectrum characterizes. Specifically, we describe the probability, p_i, of orbit members falling within a small distance, l, of a given orbit member, z_i, by defining an index $\alpha_i(l)$ through

$$p_i = l^{\alpha_i(l)}. \tag{2.2}$$

Typically, α_i takes values in a range $[\alpha_{\rm min}, \alpha_{\rm max}]$, known as the singularity spectrum. The density of singularities of type α is determined by an index $f(\alpha)$, defined by partitioning the fractal into pieces of size l and writing the number of times one finds index α' in the interval $[\alpha, \alpha + {\rm d}\alpha]$ as an expression proportional to $l^{-f(\alpha)}{\rm d}\alpha$. $f(\alpha)$ is the Hausdorff dimension of the set of singularities of strength α. $\alpha_{\rm max}$ (respectively $\alpha_{\rm min}$) is associated with the most rarefied (respectively concentrated) regions. Typically $f(\alpha_{\rm min}) = f(\alpha_{\rm max}) = 0$. Other types of singularities lie on subsets of dimension f, with $0 < f \leqslant D_0$, the Hausdorff dimension of the fractal.

To calculate the $f(\alpha)$ spectrum, we first calculate a set of generalized dimensions, D_q, related to a set of dimensions introduced by Renyi [16] (see Hentschel and Procaccia [17]).

We take a truncation of the critical point orbit $\{z_0, z_1, \ldots, z_{F_n}\}$ and form the lengths $l_i^{(n)} = |z_{i+F_{n-1}} - z_i|$, $i = 1, 2, \ldots, F_n$ with F_n being the nth Fibonacci number. These lengths serve as natural length scales for a partition with measures $p_i = 1/F_n$ associated with each length. We form the partition function

$$\Gamma_n(q, \tau) = \sum_{i=1}^{F_n} \frac{p_i^q}{(l_i^{(n)})^\tau} = \frac{1}{F_n^q} \sum_{i=1}^{F_n} (l_i^{(n)})^{-\tau} \tag{2.3}$$

mimicking the definition of Hausdorff dimension which is the case $q = 0$. $\Gamma_n = 1$ is then a simple expression for the function $q(\tau)$, from which the generalized dimensions D_q are defined by

$$\tau = (q(\tau) - 1)D_q. \tag{2.4}$$

D_0 is the Hausdorff dimension, and D_1 and D_2 are the information and correlation dimensions respectively [17].

Finally, α and $f(\alpha)$ are given from a Legendre transform of D_q:

$$\alpha(\tau) = \frac{\mathrm{d}\tau}{\mathrm{d}q} = \left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^{-1} \tag{2.5}$$

and

$$f(\alpha(\tau)) = q(\tau)\alpha(\tau) - \tau. \tag{2.6}$$

In practice, the solutions to $\Gamma_n = 1$ converge slowly with n and it is better to use the equation $\Gamma_n(q,\tau)/\Gamma_{n-1}(q,\tau) = 1$ for $q(\tau)$. Thus our approximation is

$$q(\tau) = \frac{1}{\log(F_n/F_{n-1})} \log\left(\frac{\sum_{i=1}^{F_n}(l_i^{(n)})^{-\tau}}{\sum_{i=1}^{F_{n-1}}(l_i^{(n-1)})^{-\tau}}\right). \tag{2.7}$$

We refer the reader to [1] for further details of this formalism.

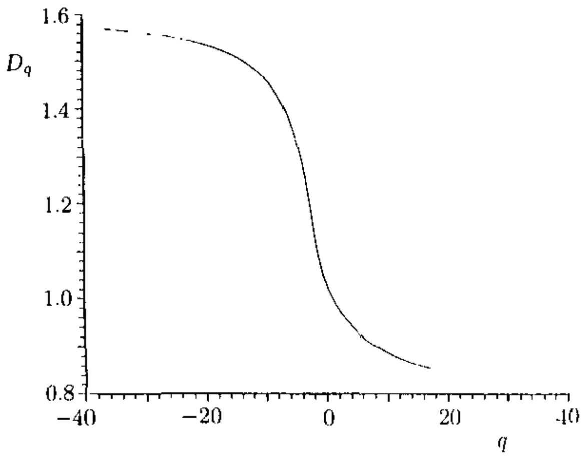

Figure 1. D_q against q for degree 2 Siegel discs.

Figure 1 shows the function D_q and figure 2 the function $f(\alpha)$ calculated with $F_n = 6765$. We have

$$\alpha_{\max} = D_{-\infty} = \frac{\log\gamma}{\log|\alpha_{\mathrm{MN}}|} \approx 1.612\,11 \tag{2.8}$$

and

$$\alpha_{\min} = D_{\infty} = \frac{\log\gamma}{\log|\alpha_{\mathrm{MN}}^2|} \approx 0.806\,06. \tag{2.9}$$

We calculate the Hausdorff dimension, D_0, for the Siegel disc boundary to be approximately 1.030.

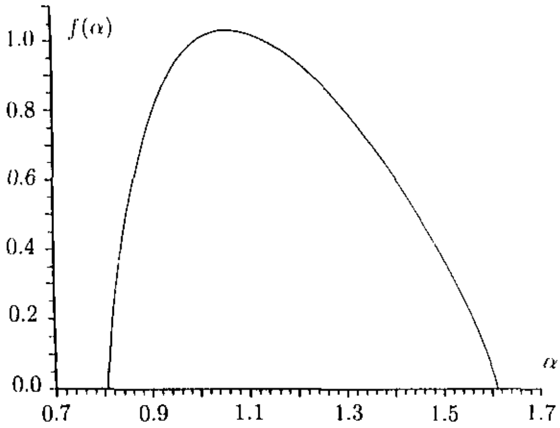

Figure 2. $f(\alpha)$ against α for degree 2 Siegel discs.

3. Degree dependence of $\alpha_{\rm MN}$

Manton and Nauenberg [7] investigated the scaling behaviour of Siegel disc boundaries in complex maps with quadratic critical points. In this section we generalize their numerical computations of golden mean scaling exponents to maps with critical points of general degree.

We consider the family of maps

$$f(z) = \frac{cz^d + b}{az^d + 1} \tag{3.1}$$

with

$$b = 1 - (a+1)(\lambda/d) \qquad c = a + (a+1)(\lambda/d)$$

and a a free real parameter which we set equal to zero for now. (We will later vary a to check universality.) These maps have the common property (even for $a \neq 0$) of having a fixed point at $z = 1$ with $f'(1) = \lambda$ and a critical point of degree d at $z = 0$. The degree need not be an integer now and $z^d = \exp(d \log z)$. There are no other finite critical points. The map (2.1) studied in section 2 is the degree 2 case of (3.1).

$\alpha_{\rm MN}$ is defined by

$$\alpha_{\rm MN} = \lim_{n \text{ even} \to \infty} \frac{z_{F_n}}{z_{F_{n-1}}}. \tag{3.2}$$

The odd limit is just $\bar{\alpha}_{\rm MN}$, where the 'bar' denotes complex conjugate.

Figure 3 shows how $\alpha_{\rm MN}$ varies in the complex plane with d and figure 4 is a plot of $|\alpha_{\rm MN}|$ against d. The graphs appears smooth and in the limit of large d we find behaviour of the form

$$\lim_{d \to \infty} |\alpha_{\rm MN}|^d \approx 0.523\,53. \tag{3.3}$$

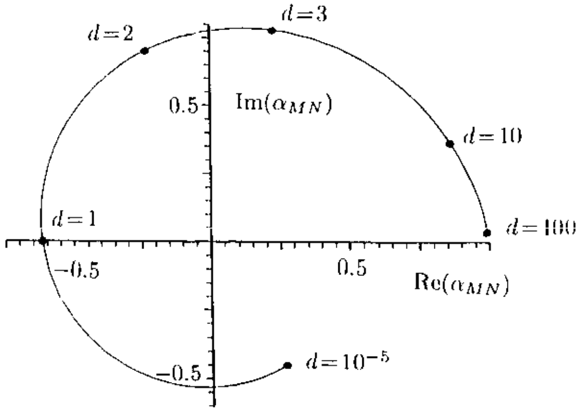

Figure 3. $\alpha_{\rm MN}$ in the complex plane as d varies.

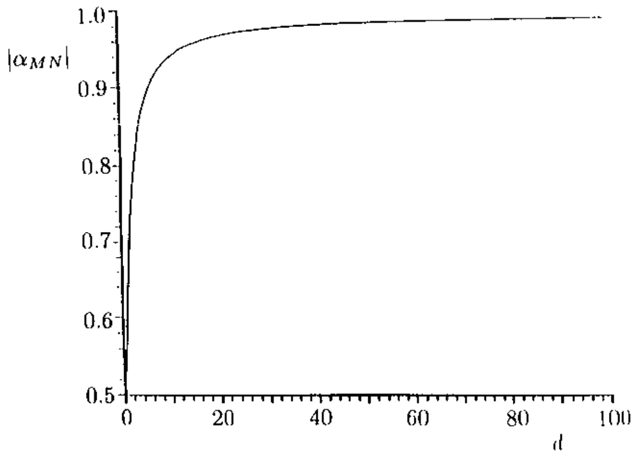

Figure 4. $|\alpha_{\rm MN}|$ against d.

We shall write this as

$$\lim_{d\to\infty} |\alpha_{\rm MN}|^d = \mathrm{e}^{-A} \tag{3.4}$$

with $A \approx 0.647\,17$, which implies

$$|\alpha_{\rm MN}| \sim 1 - (A/d) \qquad \text{as } d \to \infty. \tag{3.5}$$

See figure 5.

The behaviour as $d \to 0+$ is interesting in that, as is evident from figure 5, $\alpha_{\rm MN}$ approaches a constant with $|\alpha_{\rm MN}| \approx 0.523$. The closeness of this number to $\lim_{d\to\infty} |\alpha_{\rm MN}|^d$ in (3.3) leads one to conjecture that they may indeed be equal. A rigorous proof of (3.3) along the lines of that of Eckmann and Wittwer [11] may be possible and help explain this discovery.

By varying the parameter a in (3.1) we can check the universality of our exponents. We witness poor convergence for large d but always estimate A in (3.4) to be close to 0.65. The evidence for the universality of $\alpha_{\rm MN}$ (and hence A) seems strong though. For each of $a = 0$, 0.2, 0.5 we calculate $|\alpha_{\rm MN}| \approx 0.741\,932$, 0.812 158, 0.9376, 0.9682, 0.9936 for $d = 2$, 3, 10, 20, 100 respectively.

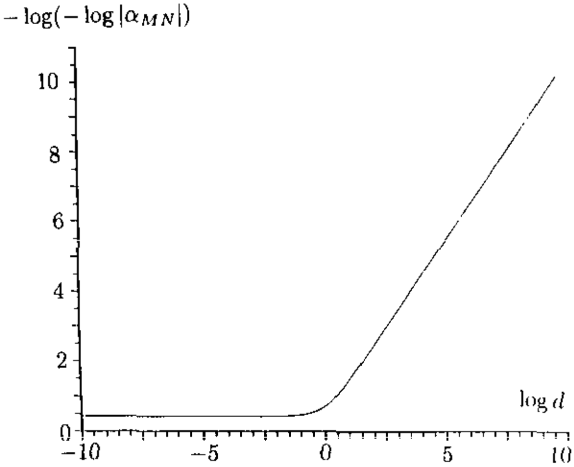

Figure 5. $-\log(-\log|\alpha_{MN}|)$ against $\log d$ showing large and small d behaviour of $|\alpha_{MN}|$.

4. Degree dependence of D_q and $f(\alpha)$

It is straightforward to generalize the calculations of section 2 for the map (3.1). Equation (2.7) is used to find first of all $q(\tau)$ and thence D_q and $f(\alpha)$. Equation (2.8) is unchanged—except that α_{MN} now depends on d—and equation (2.9) becomes

$$\alpha_{\min} = D_\infty = \frac{\log\gamma}{\log|\alpha_{MN}|^d}. \tag{4.1}$$

The asymptotic behaviour of $|\alpha_{MN}|$ (equation (3.4)) immediately leads to

$$\alpha_{\min} \sim (-\log\gamma/A) \approx 0.743\,56 \tag{4.2}$$

and

$$\alpha_{\max} \sim (-\log\gamma/A)d. \tag{4.3}$$

Figure 6 shows $f(\alpha)$ for $d = 2, 3 \ldots, 10$. Already the asymptotic behaviour is visible. The results here are similar to those obtained by van der Weele *et al* [10] for the period doubling problem. We have also attempted to calculate the degree dependence of D_0. However the convergence of our approximations for large d is very poor and we can draw no conclusions other than to say that D_0 increases monotonically as d moves away from 1.

5. Summary

We have numerically determined the $f(\alpha)$ spectrum for Siegel discs with varying degree critical points. The behaviour of the local scaling exponent α_{MN} is central to our results, and, in common with results for period doubling, the quantity $|\alpha_{MN}|^d$ is asymptotically a finite constant. Further investigation of both the limits $d \to \infty$ and

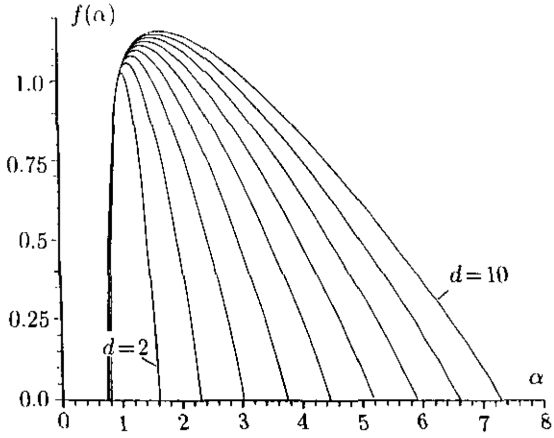

Figure 6. $f(\alpha)$ against α for $d = 2, 3, \ldots, 10$.

$d \to 0+$ by numerical and analytic means is desirable in order to explain the results found here. In particular the numerical techniques of van der Weele *et al* [10] and the analytic methods of Eckmann and Wittwer [11] should be brought to bear on the Siegel disc probelm.

References

[1] Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B I 1986 *Phys. Rev.* A **33** 1141

[2] Frisch U and Parisi G 1985 *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics. Proc. Enrico Fermi International School of Physics, Course LXXXVIII* ed M Ghil (Amsterdam: North-Holland) p 84

[3] Shenker S J 1982 *Physica* **5D** 405

[4] Jensen M H, Kadanoff L P, Libchaber A, Procaccia I and Stavans J 1985 *Phys. Rev. Lett.* **55** 2798
Gwinn E G and Westervelt R M 1987 *Phys. Rev. Lett.* **59** 157

[5] Tang C and Kohmoto M 1986 *Phys. Rev.* B **34** 2041

[6] Osbaldestin A H and Sarkis M Y 1987 *J. Phys. A: Math. Gen.* **20** L953

[7] Manton N S and Nauenberg M 1983 *Commun. Math. Phys.* **89** 555

[8] Widom M 1983 *Commun. Math. Phys.* **92** 121

[9] Hu B 1982 *Phys. Rep.* **91** 233
Hu B and Mao J-M 1982 *Phys. Rev.* A **25** 3259
Hu B and Satija I 1983 *Phys. Lett.* **98A** 143
Bhattacharjee J K and Banerjee K 1987 *J. Phys. A: Math. Gen.* **20** L269

[10] van der Weele J P, Capel H W and Kluiving R 1987 *Physica* A **145** 425

[11] Eckmann J P and Wittwer P 1985 *Computer Methods and Borel Summability Applied to Feigenbaum's Equation (Lecture Notes in Physics 227)* (Berlin: Springer)

[12] Ostlund S, Rand D, Sethna J and Siggia 1983 *Physica* **8D** 303
Hu B, Valinia A and Piro O 1990 *Phys. Lett.* **144A** 7
Delbourgo B and Kelly B G 1991 *J. Math. Phys.* **32** 1045

[13] Hu B, Shi J and Kim S-Y 1991 *Phys. Rev.* A **43** 4249

[14] Hu B and Mao J-M 1987 *J. Phys. A: Math. Gen.* **20** 1809
Ambika G and Joseph K B 1988 *J. Phys. A: Math. Gen.* **21** 3963

[15] Alstrøm P 1986 *Commun. Math. Phys.* **104** 581
Delbourgo B and Kelly B G 1991 *Phys. Rev.* A **42** 6230

[16] Renyi A 1976 *Selected Papers* vol 2 (Budapest: Akademiai Kiado) p 526

[17] Hentschel H G E and Procaccia I 1983 *Physica* **8D** 435